THE STATISTICS OF TASTE AND THE INFLIGHT ESTIMATION OF SENSOR PRECISION

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ABSTRACT

Algorithms are developed for estimating star-tracker precision in space from inflight data using the QUEST algorithm. The QUEST variable TASTE is shown to be a χ^2 variable with $2N-3$ degrees of freedom, where N is the number of stars, and its mean and variance directly related to the star-tracker measurement variance. These results are used to construct an attitude-independent estimator of star-tracker precision from inflight data. An attitude-independent estimator for the variances of multiple sensors from inflight data developed previously is reviewed and generalized to more than three sensors.

INTRODUCTION

Most attitude determination systems employing star-trackers use an attitude estimation algorithm based on the Wahba problem (ref. 1), which minimizes the least-square loss function

$$L(A) = \frac{1}{2}\sum_{i=1}^{N} a_i\,|\hat{\mathbf{W}}_i - A\hat{\mathbf{V}}_i|^2 \tag{1}$$

with attitude estimate[3] A^* given by

$$A^* = \arg\min_{A\varepsilon SO(3)} L(A) \tag{2}$$

that is, the value of the 3×3 proper orthogonal matrix A for which $L(A)$ is a minimum. Here the a_i, $i = 1, \ldots, N$, are a set of N non-negative weights, $\hat{\mathbf{W}}_i$, $i = 1, \ldots, N$, are the observed directions in the spacecraft body frame, and $\hat{\mathbf{V}}_i$, $i = 1, \ldots, N$, are the corresponding vectors in the primary reference frame (typically inertial). Generally, one assumes that the reference directions are noise-free.

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[3]More correctly *estimator*, since the $\hat{\mathbf{W}}_i$ are random vectors. When these random vectors are replaced by their actual sampled values $\hat{\mathbf{W}}_i'$, as we must in real computations, this expression yields the estimate.

A number of algorithms were proposed almost immediately for solving the Wahba problem. These and more recent approaches have been described by Markley and Mortari in their excellent review (ref. 2). Of special importance have been Davenport's q-algorithm (ref. 3) and QUEST (ref. 4), which has received wide application both for Earth-orbiting and interplanetary spacecraft. Of particular interest in the QUEST work has been the QUEST measurement model, which is

$$\hat{\mathbf{W}}_i = \hat{\mathbf{W}}_i^{\text{true}} + \Delta\hat{\mathbf{W}}_i\,, \qquad \Delta\hat{\mathbf{W}}_i \sim \mathcal{N}(\mathbf{0},\, R_i) \tag{3ab}$$

$$R_i = \sigma_i^2\left(I_{3\times3} - \hat{\mathbf{W}}^{\text{true}}\hat{\mathbf{W}}_i^{\text{true}\,T}\right), \qquad \hat{\mathbf{W}}_i^{\text{true}} = A^{\text{true}}\hat{\mathbf{V}}_i \tag{3cd}$$

The variances σ_i^2, $i = 1, \ldots, N$, are the sole parameters of the model. The individual direction measurements are assumed to have a circle of error in the tangent plane rather than the more general (and more realistic) ellipse of error. This approximation is generally adequate for focal-plane sensors with limited fields of view (such as a star tracker) but has been employed also for many sensors for which the truthfulness of its representation of sensor errors may be justifiably questioned. Nonetheless, it has led to the development of many practical attitude estimators. The assumption that $\Delta\hat{\mathbf{W}}_i$ is zero-mean and Gaussian also cannot be exactly correct (ref. 5), but is true to good approximation. Deviations in the attitude estimate due to this approximation are generally on the order of σ_i^2, which for $\sigma_i = 3$ arcsec, leads to an equivalent angle error on the order of 5×10^{-5} arcsec, surely a negligible error.

The Davenport q-algorithm (ref. 3) constructs the optimal attitude estimate $A^{*\prime}$ by first constructing the attitude profile matrix B, defined as

$$B \equiv \sum_{i=1}^{N} a_i \hat{\mathbf{W}}_i \hat{\mathbf{V}}_i^T \tag{4}$$

whence

$$L(A) = \sum_{i=1}^{N} a_i - \text{tr}[B^T A] \equiv \lambda_o - g_A(A) \tag{5}$$

with $g_A(A)$ the *gain* function. From this the following quantities are defined

$$s \equiv \text{tr}B\,, \quad S \equiv B + B' \quad \text{and} \quad \mathbf{Z} \equiv \left[B_{23} - B_{32},\, B_{31} - B_{13},\, B_{12} - B_{21}\right]^T \tag{6abc}$$

and the 4×4 matrix K

$$K = \begin{bmatrix} S - sI_{3\times3} & \mathbf{Z} \\ \mathbf{Z}^T & s \end{bmatrix} \tag{7}$$

In terms of the quaternion (ref. 6) and Davenport's matrix K the gain function may be written as

$$g_{\bar{q}}(\bar{q}) \equiv g_A(A(\bar{q})) = \bar{q}^T K \bar{q} \tag{8}$$

which is a maximum (and the loss function a minimum) for $\bar{q} = \bar{q}^*$ with

$$K\bar{q}^* = \lambda_{\max}\bar{q}^* \tag{9}$$

and $\lambda_{\max}$ is the largest eigenvalue of K. This is Davenport's q-algorithm. The earliest implementation of Davenport's q-algorithm was in SNAPLS (for SNAPshot Least-Squares), the attitude determination software system for the HEAO (ref. 7) mission (launched 1977, 1978 and 1979). The SNAPLS algorithm solved for $\bar{q}^*$ by implementing Householder's method (ref. 8) to determine the four eigenvalues and eigenvectors of K.

The QUEST algorithm (ref. 4), whose details need not concern us in the present work, offered a very fast method for computing $\lambda_{\max}$ and the associated quaternion, the QUEST measurement model (*vide supra*), the method of sequential rotations, a compact and easily calculable expression for the attitude estimate-error covariance matrix, and a very fast data validation algorithm using a variable called TASTE (hence the expression "TASTE Test"). In addition, it was also shown in ref. 4 that the cost function would be minimized for constant λ_o if one chose

$$a_i = c/\sigma_i^2 \tag{10}$$

In this case it was later shown (ref. 5) that the Wahba attitude estimate was also the maximum-likelihood estimate of the attitude given the QUEST measurement model of equations (3). Furthermore, if one chose $c = 1$, then the loss function of equation (1) became the data-dependent part of the negative-log-likelihood function (ref. 9) of the attitude given the QUEST measurement model. This gave a firm statistical basis for the Wahba problem, rather than its being purely a mathematical curiosity.

The variable TASTE, which is central to the present study, is defined as

$$\mathrm{TASTE} \equiv 2(\lambda_o - \lambda_{\max}) = 2L(A^*) \tag{11}$$

with the weights a_i being given by $1/\sigma_i^2$, that is, $c = 1$ in equation (10).

THE STATISTICS OF TASTE

Given the above choice for the weights, we write for arbitrary A

$$L(A) = \frac{1}{2}\sum_{i=1}^{N}\frac{1}{\sigma_i^2}|\hat{\mathbf{W}}_i - A\hat{\mathbf{V}}_i|^2 \tag{12}$$

Note that in equation (12) the $\hat{\mathbf{W}}_i$ and hence $\lambda_{\max}$ are random variables. Note also that A^{true} minimizes

$$L^{\mathrm{true}}(A) \equiv \frac{1}{2}\sum_{i=1}^{N}\frac{1}{\sigma_i^2}|\hat{\mathbf{W}}_i^{\mathrm{true}} - A\hat{\mathbf{V}}_i|^2 \tag{13}$$

and

$$\lambda_{\max}^{\mathrm{true}} = \lambda_o = \frac{1}{\sigma_{\mathrm{tot}}^2} \equiv \sum_{i=1}^{N}\frac{1}{\sigma_i^2} \tag{14}$$

We define now the attitude increment vector ξ according to

$$A = A(\xi) \equiv \delta A(\xi) A^{\mathrm{true}} \tag{15}$$

with

$$\begin{aligned}\delta A(\xi) \equiv \exp\{\,[[\,\xi\,]]\,\} &= I_{3\times3} + \frac{\sin|\xi|}{|\xi|}[[\,\xi\,]] + \frac{1-\cos|\xi|}{|\xi|^2}[[\,\xi\,]]^2 \\ &= I_{3\times3} + [[\,\xi\,]] + O(|\xi|^2)\end{aligned} \tag{16}$$

and (ref. 6)

$$[[\,\mathbf{u}\,]] \equiv \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{bmatrix} \tag{17}$$

The matrix $\delta A(\xi)$ defined by equation (16) is exactly proper orthogonal. We anticipate that ξ^* will be on the order of σ_{tot} with very large probability. The attitude increment vector is clearly a rotation vector (ref. 6). Substituting now equation (15) into equation (12) we obtain after some manipulation[4]

$$L_\xi(\xi) \equiv L(A(\xi)) = \frac{1}{2}\sum_{i=1}^{N}\frac{1}{\sigma_i^2}\,\left|\,\Delta\hat{\mathbf{W}}_i + [[\,\hat{\mathbf{W}}_i^{\text{true}}\,]]\,\xi\,\right|^2 \tag{18}$$

Let $S(\hat{\mathbf{W}}_i^{\text{true}})$ be any constant proper orthogonal matrix which accomplishes the transformation

$$S(\hat{\mathbf{W}}_i^{\text{true}})\,\hat{\mathbf{W}}_i^{\text{true}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \equiv \hat{\mathbf{3}} \tag{19}$$

whence

$$S(\hat{\mathbf{W}}_i^{\text{true}})\,R_i\,S^T(\hat{\mathbf{W}}_i^{\text{true}}) = \sigma_i^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{20}$$

Defining

$$\hat{\mathbf{U}}_i \equiv \frac{1}{\sigma_i}\,S(\hat{\mathbf{W}}^{\text{true}})\hat{\mathbf{W}}_i \tag{21}$$

we can write

$$2L_\xi(\xi) = \sum_{i=1}^{N}\left|\,\Delta\hat{\mathbf{U}}_i + [[\,\hat{\mathbf{U}}^{\text{true}}\,]]\,\xi\,\right|^2 \tag{22}$$

Defining further in obvious notation

$$\Delta u_{2i-1} \equiv \hat{\mathbf{1}}^T\Delta\hat{\mathbf{U}}_i \quad \text{and} \quad \Delta u_{2i} \equiv \hat{\mathbf{2}}^T\Delta\hat{\mathbf{U}}_i\,, \qquad i = 1,\,\ldots\,,\,N \tag{23ab}$$

the projections of $\hat{\mathbf{U}}_i$ along the two axes perpendicular to $\hat{\mathbf{3}}$, respectively, and similarly

$$H_{2i-1} \equiv -\hat{\mathbf{1}}^T[[\,\hat{\mathbf{U}}_i^{\text{true}}\,]] \quad \text{and} \quad H_{2i} \equiv -\hat{\mathbf{2}}^T[[\,\hat{\mathbf{U}}_i^{\text{true}}\,]] \tag{24ab}$$

we may write the loss function as

$$2L_\xi(\xi) = \sum_{i=1}^{2N}|\Delta u_i - H_i\,\xi|^2 \tag{25}$$

The third component of $\hat{\mathbf{U}}_i$ does not contribute to the loss function, since in our measurement model it is identically zero and its sensitivity to ξ also vanishes by explicit construction. Note in particular that the $2N$ effective scalar measurement noise terms are independent and satisfy

$$\Delta u_i \sim \mathcal{N}(0,\,1)\,, \qquad i = 1,\,\ldots\,,\,2N \tag{26}$$

[4] If we calculate the difference in equation (13) before computing the square, then the $O(|\xi|^2)$ term of equation (16) would have led to a term of order $O(|\xi|^3)$ in our evaluation of equation (18), which we discard.

The maximum-likelihood estimate of ξ is trivially

$$\xi^* = P \sum_{i=1}^{2N} H_i^T \Delta u_i \quad \text{with} \quad P^{-1} = \sum_{i=1}^{2N} H_i^T H_i \tag{27ab}$$

and

$$\xi^* \sim \mathcal{N}(\mathbf{0},\ P) \tag{28}$$

Writing

$$2L_\xi(\xi) = \sum_{i=1}^{2N} |(\Delta u_i - H_i\, \xi^*) - H_i\, (\xi - \xi^*)|^2 \tag{29}$$

we obtain straightforwardly

$$2L_\xi(\xi^{\text{true}}) = 2L(\xi^*) + (\xi^* - \xi^{\text{true}})^T P^{-1} (\xi^* - \xi^{\text{true}}) \tag{30}$$

The absence of a cross term is a natural consequence of the Luenberger projection theorem (ref. 10), but we obtain it equally well in this trivial example by direct calculation.

By definition

$$\xi^{\text{true}} = \mathbf{0} \tag{31}$$

and from equation (25)

$$2L(A^{\text{true}}) = \sum_{i=1}^{2N} |\Delta u_i|^2 \sim \chi^2(2N) \tag{32}$$

a χ^2 random variable with $2N$ degrees of freedom. Clearly, from equations (28) and (30) one has that

$$(\xi^* - \xi^{\text{true}})^T P^{-1} (\xi^* - \xi^{\text{true}}) \sim \chi^2(3) \tag{33}$$

Thus,

$$\chi^2(2N) = \text{TASTE} + \chi(3) \tag{34}$$

Since the two terms in the right member of equation (34) are statistically independent, then by Cochran's theorem or, equivalently, Fisher's theorem (both ref. 11) equation (34) can be true only if[5]

$$\text{TASTE} \sim \chi^2(2N - 3) \tag{35}$$

As an immediate consequence of equation (35) it follows that

$$E\{\,\text{TASTE}\,\} = 2N - 3 \qquad \text{and} \qquad \text{Var}\{\,\text{TASTE}\,\} = 2(2N - 3) \tag{36}$$

for the expectation and the variance of TASTE.

THE PRECISION ESTIMATOR

The most obvious method for constructing an estimator for star-tracker precision is by maximum-likelihood estimation. Keeping also model terms, the negative-log-likelihood function of the attitude

[5]This simple result has been known to the first author since 1981 but was never published, except as a comment in the updated original FORTRAN code for QUEST and in a internal report in 1993. That author's unpublished result was quoted by Markley and Mortari (ref. 2) without proof or attribution.

given the measurements and the value of σ^2 the common variance for star-tracker direction measurements may be written

$$J(A|\sigma^2) = \frac{1}{2}\left\{\frac{1}{\sigma^2}\,\tilde{L}(A) + 2N\log\sigma^2 + 2\log 2\pi + 2\log f\right\} \tag{37}$$

where

$$\tilde{L}(A) \equiv \sum_{i=1}^{N} |\hat{\mathbf{W}}_i - A\hat{\mathbf{V}}_i|^2 \tag{38}$$

Note the factors of 2 rather than 3 appear in equation (37). because only two of the components of $\hat{\mathbf{W}}_i$ have statistical significance. The term f is a correction term for the finiteness of the unit sphere (ref. 5) (a Gaussian distribution extends to infinity) and may be neglected, even though it depends on σ. The maximum-likelihood estimate of σ^2 will minimize $J(A^*|\sigma^2)$, hence it must satisfy

$$\left.\frac{\partial}{\partial\sigma^2}\,J(A^*|\sigma^2)\right|_{(\sigma^2)^*} = 0 \tag{39}$$

Since A^* does not depend on σ^2, this leads straightforwardly to

$$-\frac{1}{\left((\sigma^2)^*\right)^2}\,\tilde{L}(A^*) + \frac{2N}{(\sigma^2)^*} = 0 \tag{40}$$

or

$$(\sigma^2)^* = \frac{1}{2N}\,\tilde{L}(A^*) \tag{41}$$

From the known statistical properties of TASTE, it follows that

$$E\{\,(\sigma^2)^*\} = \frac{2N-3}{2N}\,(\sigma^{\text{true}})^2 \tag{42}$$

so that, as is usually the case for maximum-likelihood estimation of a variance, the estimator is only asymptotically consistent (that is, as $N \to \infty$). One usually eliminates this embarrassment by simply multiplying the estimator by a factor $2N/(2N-3)$. Thus, a consistent estimator of σ^2 is simply

$$(\sigma^2)^* = \frac{1}{2N-3}\,\tilde{L}(A^*) \tag{43}$$

which is certainly a valid estimator if not a maximum-likelihood estimator.

The derivation up to now has assumed a single-frame of simultaneous data. In the case where one has n frames of (simultaneous) data, each with N_k star-direction measurements, one write the σ^2 estimator as

$$(\sigma^2)^* = \frac{1}{2N_{\text{tot}} - 3n}\,\sum_{k=1}^{n}\tilde{L}_k(A_k^*) \tag{44}$$

with

$$N_{\text{tot}} \equiv \sum_{k=1}^{n} N_k \tag{45}$$

Note that for $\tilde{L}_k(A_k)$

$$a_i = 2, \quad i = 1, \ldots, N_k \qquad \text{and} \qquad \lambda_{o,k} = 2N_k \tag{46}$$

Hence, one can write the σ^2 estimator equivalently as

$$(\sigma^2)^* = \frac{1}{2N_{\rm tot} - 3n} \sum_{k=1}^{n} (2N_k - \lambda_{{\rm max},k}) \tag{47}$$

This estimator is obviously consistent and has variance

$$\mathrm{Var}\{\, (\sigma^2)^* \,\} = \frac{2}{2N_{\rm tot} - 3n} \, (\sigma^{\rm true})^4 \tag{48}$$

whence,

$$\sigma_{\sigma^*} = \frac{1}{2}\sqrt{\frac{2}{2N_{\rm tot} - 3n}} \; \sigma^{\rm true} \tag{49}$$

with $\sigma^* \equiv \sqrt{(\sigma^2)^*}$.

APPLICATION TO AUTONOMOUS STAR TRACKERS

Additional steps may be required for application of these algorithms ot autonomous star trackers.[6] Currently, autonomous star trackers generally output the QUEST attitude quaternion, the number of stars, and the star positions. Some may output as well the attitude-error covariance matrix computed using the QUEST formula

$$P_{\theta\theta}^{-1} = \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \left(I - \hat{\mathbf{W}}_i \hat{\mathbf{W}}_i^T \right) \tag{50}$$

where the vendor will have used a universal value $\sigma_{\rm vend}$ for σ_i in his software. None of these star trackers, however, output $\lambda_{\rm max}$ as well.

For the first star-trackers one has no choice but to compute $\lambda_{\rm max}$ from the individual star positions, a laborious process if one wished to implement the TASTE test in real time. For the second, however, we can compute $\lambda_{\rm max}$ from (ref. 1)

$$\lambda_{\rm max}^{\rm vend} = \frac{1}{2} \, \mathrm{tr}(P_{\theta\theta}^{\rm vend})^{-1} \tag{51}$$

where "tr" denotes the trace operation.[7] $\lambda_o^{\rm vend}$ is then given by

$$\lambda_o^{\rm vend} = \frac{N}{\sigma_{\rm vend}^2} \tag{52}$$

Equation (51) yields $\lambda_{\rm max}$, because it uses the observed positions of the stars in the star-tracker frame. If it used the true positions, the result would have been λ_o.

The estimation of σ^2 can be obtained from

$$\mathrm{TASTE}^{\rm vend} = 2(\lambda_{\rm max}^{\rm vend} - \lambda_o^{\rm vend}) \tag{53}$$

[6]The authors are grateful to Dr. David R. Haley for making them aware of these special needs for autonomous star trackers.

[7]Note that one must have the entire $(P_{\theta\theta}^{\rm vend})^{-1}$ in order to calculate both $\lambda_{\rm max}^{\rm vend}$ and $P_{\theta\theta}^{\rm vend}$ (or even the diagonal elements of the latter). $\lambda_{\rm max}^{\rm vend}$ cannot be calculated knowing only the diagonal elements of $P_{\theta\theta}^{\rm vend}$, because the cancellations in equation (53) below are very sensitive and can easily result in the loss of ten significant figures.

The derivation of the estimator is straightforward and leads to

$$(\sigma^2)^* = \frac{\sigma^2_{\text{vend}}}{2N_{\text{tot}} - 3n} \sum_{k=1}^{n} \text{TASTE}_k^{\text{vend}} \tag{54}$$

with

$$E\{(\sigma^2)^*\} = \sigma^2 \qquad \text{and} \qquad \text{Var}\{(\sigma^2)^*\} = 2\,\frac{\sigma^4}{2N_{\text{tot}} - 3n} \tag{55}$$

NUMERICAL EXAMPLE

To test the algorithm we have simulated 100 frames of star-tracker data, each with 6 star-direction measurements. Thus, for this data set $2N_{\text{tot}} - 3n = 900$. We have chosen $\sigma^{\text{true}} = 3$ arcsec, typical of modern CCD star-trackers. Thus, we anticipate a standard deviation of the estimate of the star-tracker variance of $\sigma_{\sigma_{\sigma^*}} = \sigma^{\text{true}}/\sqrt{1800} \approx .071$ arcsec, and the estimator for such a data set should satisfy

$$\sigma^* \approx 3.0000 \text{ arcsec} \pm 0.0707 \text{ arcsec} \equiv \sigma^{\text{true}} \pm \sigma_{\sigma^*} \tag{56}$$

and have a Gaussian distribution according to the central limit theorem.

To test this, 160,000 trials of each test were simulated and the estimate of σ computed for each trial. Calculating the statistics of the samples we anticipate

$$\mu_{\sigma^*}^{\text{sampled}} = \sigma^{\text{true}} \pm \sigma_{\sigma_{\sigma^*}}/\sqrt{N} \quad = 3.0 \text{ arcsec} \pm 0.00035 \text{ arcsec} \tag{57a}$$

$$\sigma_{\sigma^*}^{\text{sampled}} = \sigma_{\sigma_{\sigma^*}} \pm \sqrt{2/N}\,\sigma_{\sigma_{\sigma^*}} \quad = 0.0707 \text{ arcsec} \pm .0005 \text{ arcsec} \tag{57b}$$

The result of the sampling was

$$\mu_{\sigma^*}^{\text{sampled}} = 2.99199 \text{ arcsec} \qquad \text{and} \qquad \sigma_{\sigma^*}^{\text{sampled}} = 0.07089 \text{ arcsec} \tag{58ab}$$

In good absolute agreement with the expected results. However, we note that the agreement of the sampled mean value of σ^* over the 160,000 trials agrees with the true mean to within 0.0080 arcsec or $\sigma_{\sigma^*}/9$, which is adequate agreement from a practical standpoint. However, we had anticipated a disagreement on the order of $\sigma_{\sigma^*}/200$. (For 10,000 samples of σ^*, the sampled mean was 0.299220.) This discrepancy is being investigated.

VARIANCE ESTIMATORS FOR MULTIPLE SENSORS

The estimation of star-tracker precision from star-tracker data alone was possible, because typically the star-tracker yields simultaneous measurements of more than one star. This is not the case for most sensors, such as a Sun sensor, an Earth horizon sensor, or a vector magnetometer. There is only one Sun, one Earth, and one geomagnetic field. For these sensors a different strategy is needed. In fact, such a strategy was developed more than two decades ago (ref. 12). For completeness, and because this estimator has much in common with the algorithm for estimating star-tracker precision, it will be reviewed briefly here.[8] This will also give us the opportunity to present the mathematics of that algorithm is somewhat more detail.

[8] The reader is cautioned to avoid the treatment of inflight sensor alignment estimation in ref. 11. That work was very primitive and has been superceded by more mature work for both batch (refs. 13, 14) and Kalman filter (ref. 15) implementations.

Suppose the spacecraft is provided with M single-direction sensors and consider the pair-wise cost function

$$\tilde{L}_{ij}(A) = \frac{1}{2}\left\{a_i|\hat{\mathbf{W}}_i - A\hat{\mathbf{V}}_i|^2 + a_j|\hat{\mathbf{W}}_j - A\hat{\mathbf{V}}_j|^2\right\} \quad i \neq j = 1, \ldots, M \tag{59}$$

It follows (ref. 4) that

$$\begin{aligned}\lambda_{\max,\,ij} &= \lambda_{o,\,ij} - \tilde{L}_{ij}(A^*_{ij}) \\ &= \sqrt{a_i^2 + 2a_i a_j \cos(\theta_{W,\,ij} - \theta_{V,\,ij}) + a_2^2}\end{aligned} \tag{60}$$

where

$$\cos(\theta_{W,\,ij} - \theta_{V,\,ij}) \equiv (\hat{\mathbf{W}}_i \cdot \hat{\mathbf{W}}_j)(\hat{\mathbf{V}}_i \cdot \hat{\mathbf{V}}_j) + |\hat{\mathbf{W}}_i \times \hat{\mathbf{W}}_j|\,|\hat{\mathbf{V}}_i \times \hat{\mathbf{V}}_j| \tag{61}$$

The quantity $\theta_{W,\,ij}$ is the angle between $\hat{\mathbf{W}}_i$ and $\hat{\mathbf{W}}_j$, while $\theta_{V,\,ij}$ is the angle between $\hat{\mathbf{V}}_i$ and $\hat{\mathbf{V}}_j$. Clearly, the latter is equal to the angle between $\hat{\mathbf{W}}_i^{\text{true}}$ and $\hat{\mathbf{W}}_j^{\text{true}}$, which we write as $\theta_{W,\,ij}^{\text{true}}$. Thus, we write

$$\Delta\theta_{ij} \equiv \theta_{W,\,ij} - \theta_{W,\,ij}^{\text{true}} \tag{62}$$

and, equivalent to equation (60),

$$\begin{aligned}\lambda_{\max,\,ij} &= \sqrt{\lambda_{o,\,ij}^2 - 4a_i a_j \sin^2(\Delta\theta_{ij}/2)} \\ &= \lambda_{o,\,ij} - \frac{a_i a_j}{2(a_i + a_j)}\,\Delta\theta_{ij}^2 + O(|\Delta\theta_{ij}|^4)\end{aligned} \tag{63}$$

and we define as the elemental effective measurement

$$z_{ij} \equiv \frac{2(a_i + a_j)}{a_i a_j}\,(\lambda_{o,ij} - \lambda_{\max,\,ij}) = \Delta\theta_{ij}^2 \tag{64}$$

The similarity to the star-tracker precision estimator $(\sigma^2)^*$ is evident. Our estimation problem reduces largely to determining the statistical properties of $\Delta\theta_{ij}^2$.

Since $\Delta\theta_{ij}$ is an infinitesimal quantity, we can write to $O(|\Delta\theta_{ij}|^3)$

$$\Delta\theta_{ij} \approx \sin(\Delta\theta_{ij}) = \Delta|\hat{\mathbf{W}}_i \times \hat{\mathbf{W}}_j| = \hat{\mathbf{s}}_{ij}^T\,\Delta(\hat{\mathbf{W}}_i \times \hat{\mathbf{W}}_j) \tag{65}$$

with

$$\hat{\mathbf{s}}_{ij} \equiv (\hat{\mathbf{W}}_i^{\text{true}} \times \hat{\mathbf{W}}_j^{\text{true}})/|\hat{\mathbf{W}}_i^{\text{true}} \times \hat{\mathbf{W}}_j^{\text{true}}| \tag{66}$$

the unit vector normal to the plane of $\hat{\mathbf{W}}_i^{\text{true}}$ and $\hat{\mathbf{W}}_j^{\text{true}}$. From equation (3a) this becomes

$$\Delta\theta_{ij} = (\hat{\mathbf{W}}_j^{\text{true}} \times \hat{\mathbf{s}}_{ij})^T\,\Delta\hat{\mathbf{W}}_i - (\hat{\mathbf{W}}_i^{\text{true}} \times \hat{\mathbf{s}}_{ij})^T\,\Delta\hat{\mathbf{W}}_j \tag{67}$$

Equation (67) makes the statement that only the components of $\Delta\hat{\mathbf{W}}_i$ and $\Delta\hat{\mathbf{W}}_j$ in the plane of $\hat{\mathbf{W}}_i^{\text{true}}$ and $\hat{\mathbf{W}}_j^{\text{true}}$ contribute to $\Delta\theta_{ij}$. The calculation of the statistical moments of $\Delta\theta_{ij}$ is now straightforward and yield (for i, j, m and p distinct indices)

$$E\{z_{ij}\} = \sigma_i^2 + \sigma_j^2 \tag{68a}$$

$$\text{Var}\{\,z_{ij}\,\} = 2(\sigma_i^2 + \sigma_j^2)^2 \tag{68b}$$

$$\text{Covar}\{z_{ij},\, z_{im}\} = 2\sigma_i^4\,|\hat{\mathbf{s}}_{ij} \cdot \hat{\mathbf{s}}_{im}|^2 \tag{68c}$$

$$\text{Covar}\{z_{ij},\, z_{mp}\} = 0 \tag{68d}$$

For the direction data at time t_k we can define a measurement vector

$$\mathbf{Z}_k = [z_{12,k}, z_{13,k}, \ldots, z_{1n,k}, z_{23,k}]^T \tag{69}$$

a parameter vector $\boldsymbol{\Sigma}$

$$\boldsymbol{\Sigma} \equiv [\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2]^T \tag{70}$$

and a measurement sensitivity matrix H_k

$$H_k \equiv [H_{12,k}^T \; H_{13,k}^T \; \cdots \; H_{1n,k}^T \; H_{23,k}^T]^T \tag{71}$$

where $H_{ij,k}$ is a $1 \times (n+1)$ array in which all elements vanish except for the ith and jth, which are unity. Given these quantities the measurement equation for the variances at time t_k becomes

$$\mathbf{Z}_k = H_k \boldsymbol{\Sigma} + \Delta\mathbf{Z}_k \tag{72}$$

where $\Delta\mathbf{Z}_k$ is zero mean and with covariance matrix easily constructed from equations (68). The estimation of the sensor variances is now a trivial exercise in least-square estimation.

The above algorithm was actually applied to real data. In 1981, reanalysis of attitude data for the Magsat mission (launched October 1979) led to the suspicion that the fine attitude sensors (two Ball Brothers CT-401 fixed-head star trackers and an Adcole precise Sun sensor, all with precisions in the neighborhood of 13 arcsec) might have larger error levels than had been claimed by the vendors. The above algorithm, however, indicated that the three sensors all performed within the vendors' specifications. The problem was later determined to arise from a software error. Reference 11 contains more details of the specific application and numerical results.

RECOMMENDATIONS FOR MANUFACTURERS OF AUTONOMOUS STAR TRACKERS

Based on our results above, we recommend that manufacturers of autonomous star trackers include not only the estimated attitude quaternion in their outputs, but the full *inverse* covariance matrix as well, since this contains much additional information, not only for calculating TASTE but also for understanding the correlations in the attitude quaternion errors, which may be high in cases of poor observability. The inverse covariance matrix is preferable to the covariance matrix, since the inverse covariance matrix is the more fundamental quantity, and the inversion of the matrix onboard may result in loss of significance. Output of only the diagonal elements of the covariance matrix or inverse covariance matrix is not adequate for post-launch analysis. It would also be a good idea to include TASTE and σ_{vend} among the outputs also, since the latter quantity may be different from star tracker to star tracker (as technology improves) and the manufacturer's specification literature may not always include the latest value, especially if units are individually calibrated before delivery. Needless to say, the number of stars observed for each quaternion calculation, should also be output, which seems to be currently the case.

DISCUSSION

While the algorithms presented in the paper for the estimation of precision estimation perform well, the simulation results, naturally, are unrealistically good, because they do not take into account sensor and environmental modeling errors, which are not expected to be zero-mean or Gaussian. In addition, the estimated σ's will also include random errors in the environmental models. However, even if the estimated precisions do not reflect the true instrumental precision of the star tracker, they do provide an effective sensor variance which can used to establish the performance of the instrument and the resulting attitude estimator. In a more realistic setting one would estimate not only sensor error-level parameters but calibration parameters as well. The present treatment assumes, effectively, that calibration parameters have already been estimated with an accuracy much better than that of the sensor error level.

Current researches are directed to estimating more realistic noise parameters.

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